THE NATURAL FORCE DENSITY METHOD FOR THE SHAPE FINDING OF MEMBRANE STRUCTURES

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Summary. This paper discusses the *natural force density method* (NFDM), an extension of the well known *force density method* for the shape finding of continuous membrane structures, which preserves the linearity of the original method. Besides, if the NFDM is applied iteratively, it converges to a configuration under a uniform and isotropic plane stress field. This means that a minimal surface for a membrane can be achieved through a succession of viable configurations, in such a way that the process can be stopped at any iteration, and the result assumed as good. The NFDM can also be employed to the shape finding of non-minimal surfaces, in which cases there is no guarantee that a prescribed, non-isotropic stress field can be achieved through iterations. The paper presents several examples of application of the NFDM to the shape finding of minimal and non-minimal membrane surfaces.

Resumo. *Este trabalho discute o método* da densidade natural de forças *(NFDM), uma extensão do conhecido* método das densidades de força *para a busca da forma de estruturas contínuas de membrana, o qual preserva a linearidade característica do método original. Além disso, se for aplicado iterativamente, o NFDM converge para uma configuração sob um campo uniforme e isótropo de tensões. Isto significa que uma configuração minimal pode ser alcançada através de uma sucessão de configurações viáveis, de modo que o processo pode ser interrompido em qualquer iteração. O NFDM também pode ser aplicado para a busca da forma de superfícies não-minimais, porém neste caso não há garantia de que um campo de tensões arbitrariamente imposto possa ser atingido. O trabalho apresenta vários exemplos de aplicação do NFDM à busca da forma de superfícies de membranas minimais e não-minimais.*

Resumen. *Este En este trabajo se discute el método de densidad natural de fuerzas (NFDM), una extensión del bien conocido método de densidad de fuerzas para a búsqueda de la forma en estructuras textiles, la cual preserva la linealidad característica del método original . Por otro lado, si el NFDM se aplica de forma interactiva el va a converger a una configuración dentro un campo de tensiones isotrópico y uniforme. Esto significa que se puede lograr una superficie mínima para una membrana a través de la sucesión de configuraciones viables, de tal manera que el proceso puede pararse en cualquier interacción y asumir el resultado como correcto. El NFDM puede también utilizarse para la búsqueda de la forma de superficies no mínimas, pero en este caso no se garantiza de que el campo de tensiones impuesto arbitrariamente pueda ser logrado. En este trabajo se presentan varios ejemplos de aplicaciones del método NFDM en la búsqueda de la forma de superficies mínimas y no mínimas.*

1 INTRODUCTION

Design and analysis of membrane structures constitute an integrate process, including procedures for shape finding, patterning and load analysis. The Finite Element Method is a versatile way to pose this overall process, directly providing, besides a viable shape, also a map of the stresses to which the structure is subjected. It is also adequate to determine the behavior of the structure under design loads, as well as to transfer data to the patterning routines. On the other hand, procedures based on the FEM or in other forms of structural analysis result in nonlinear analyses, and require specification of a convenient initial geometry, load steps and boundary conditions, which are not always known from start.

An alternative method for finding viable configurations, which avoids the problems related to nonlinear analysis, is given by the *force density method* (FDM), which was first proposed in the context of cable nets[1,2,3]. The method is routinely applied to shape finding of membrane surfaces, replacing the membrane by an equivalent cable network, which must be as regular as possible (otherwise is may become quite dubious which force densities should be prescribed to achieve a desired shape).

This paper discusses an extension of the force density method for the shape finding of continuous membrane structures, which preserves the linearity of the original method. The new NFDM was first suggested by Pauletti[4], based on the natural approach introduced by Argyris[5] for the Finite Element Method. Afterwards, Pauletti[6] presented a more rigorous foundation for the method, recognizing that the imposition of natural force densities (NFD) is equivalent to the imposition of 2nd Piola-Kirchhoff stresses to a reference mesh, a property first recognized by Bletzinger[7] for the original force density method.

2 FORMULATION OF THE NATURAL FORCE DENSITY METHOD

Consider the three-node plane triangular finite element shown in Figure 1. Let l_i^0, l_i^r and l_i, $i = 1,2,3$, be the element side lengths at an undeformed, a reference and an equilibrium configurations, respectively. We define three "natural deformations" along the sides of the element, according to $\varepsilon_i = \left(l_i - l_i^0\right)\left(l_i^0\right)^{-1}$, and collect them in a *vector of natural deformations* $\boldsymbol{\varepsilon}_n = \begin{bmatrix}\varepsilon_1 & \varepsilon_2 & \varepsilon_3\end{bmatrix}^T$. There exists a linear relationship between $\boldsymbol{\varepsilon}_n$ and the linear Green strains, $\boldsymbol{\varepsilon}_n = \mathbf{T}\boldsymbol{\varepsilon}$, from which we can define a *vector of natural stresses* $\boldsymbol{\sigma}_n = \mathbf{T}^{-T}\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ is the vector of Cauchy stresses acting on the element. It can be shown that $\boldsymbol{\sigma}_n$ and $\boldsymbol{\varepsilon}_n$ are energetically conjugate.

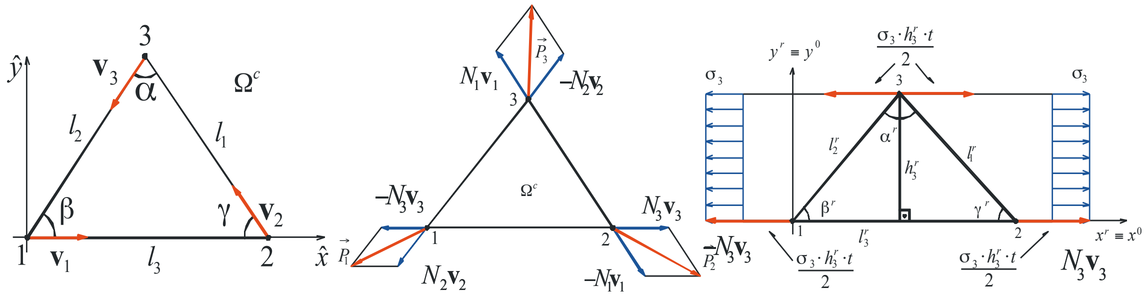

Figure 1. *(a)* Unit vectors $\mathbf{v}_i$, $i = 1,2,3$, along the element edges; *(b)* internal nodal forces $\mathbf{p}_i$, decomposed into natural forces $N_i\,\mathbf{v}_i$; *(c)* determination of natural force N_3.

We also define three "natural forces" N_i acting along the sides of the element, according to $N_i = V l_i^{-1} \sigma_i$, where V is the volume of the element, and collect them into a *natural force vector* $\mathbf{p}_n = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}^T$. Furthermore, we define the *vector of the natural force densities* according to $\mathbf{n} = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix}^T = \mathcal{L}^{-1}\mathbf{p}_n = V\mathcal{L}^{-2}\mathbf{T}^{-T}\boldsymbol{\sigma}$, where $\mathcal{L} = \operatorname{diag}\left\{ l_1 \quad l_2 \quad l_3 \right\}$.

Thereafter, we show that, for a prescribed vector $\mathbf{n}$, there is a linear relationship between the natural force vector $\mathbf{p}_n$ and element nodal coordinates $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix}^T$, according to $\mathbf{P}_n = \mathbf{k}_n \mathbf{x}$, where $\mathbf{x}_i, i = 1,2,3$, are the position vectors of the element nodes at the equilibrium configuration and $\mathbf{k}_n$ is a constant *element stiffness matrix* given by

$$\mathbf{k}_n = \begin{bmatrix} (n_2 + n_3)\mathbf{I} & -n_3\mathbf{I} & -n_2\mathbf{I} \\ \cdot & (n_1 + n_3)\mathbf{I} & -n_1\mathbf{I} \\ sym. & \cdot & (n_1 + n_2)\mathbf{I} \end{bmatrix}. \qquad (1)$$

Thus, after assembling the load and stiffness contributions of all elements, we arrive to a linear problem at the structural level, which is completely independent of any reference configuration. However, instead of prescribing directly some $\mathbf{n}$ for each element, it may be more convenient calculate them from stresses $\boldsymbol{\sigma}_r$ defined in a reference configuration, according to $\mathbf{n}_r = V^r \mathcal{L}_r^{-2} \mathbf{T}_r^{-T} \boldsymbol{\sigma}_r$. It can be shown that $\boldsymbol{\sigma}_r$ corresponds to the 2nd Piola-Kirchhoff stresses (2PK) associated to the final Cauchy stresses, calculated at the equilibrium configuration according to $\boldsymbol{\sigma} = \left(V^{-1} \mathcal{L}^2 \mathbf{T} \right) \mathbf{n}_r$.

If the NFDM is applied iteratively, always re-imposing a constant, uniform and isotropic 2PK stress field, the method will converge to a configuration under a uniform, isotropic Cauchy stress field. This means that a minimal surface for a membrane can be achieved through a succession of viable configurations, in such a way that the process can be stopped at any iteration, and the result assumed as good. This is a clear advantage, if compared to Newton's Method, which may also converge to a minimal solution, but through a series of unfeasible, non-equilibrium configurations.

Moreover, the NFDM can also be applied to the shape finding of non-minimal membrane surfaces, through the imposition of non-isotropic 2PK stress fields. In this case, however, even though a viable shape can still be obtained at every linear step, there is no guarantee that an arbitrary prescribed, non-isotropic Cauchy stress field can be achieved through iterations. Furthermore, since geometry varies during iterations, definition of principal stress directions becomes more complicate.

3 APPLICATIONS

As an application of the linear NFDM, consider the transformation of the same square reference mesh into different surfaces, in a single NFDM step. The first row of Fig. 2 shows the same reference mesh transformed into different shapes, simply prescribing displacements to some selected nodes, along with a uniform 2PK stress field on the membrane and normal loads on the border cables. Of course, Cauchy stresses at the equilibrium configurations are no longer uniform. This is fully coherent with the original FDM, where final which also has no control over the normal forces acting on cables, at the equilibrium configuration.

Although the original mesh geometry is basically irrelevant, the *topological genus* of the surface has to be respected. Thus, in order the produce a conoidal surface, a hole must be cut into the original mesh, as shown in the second line of Fig. 2.

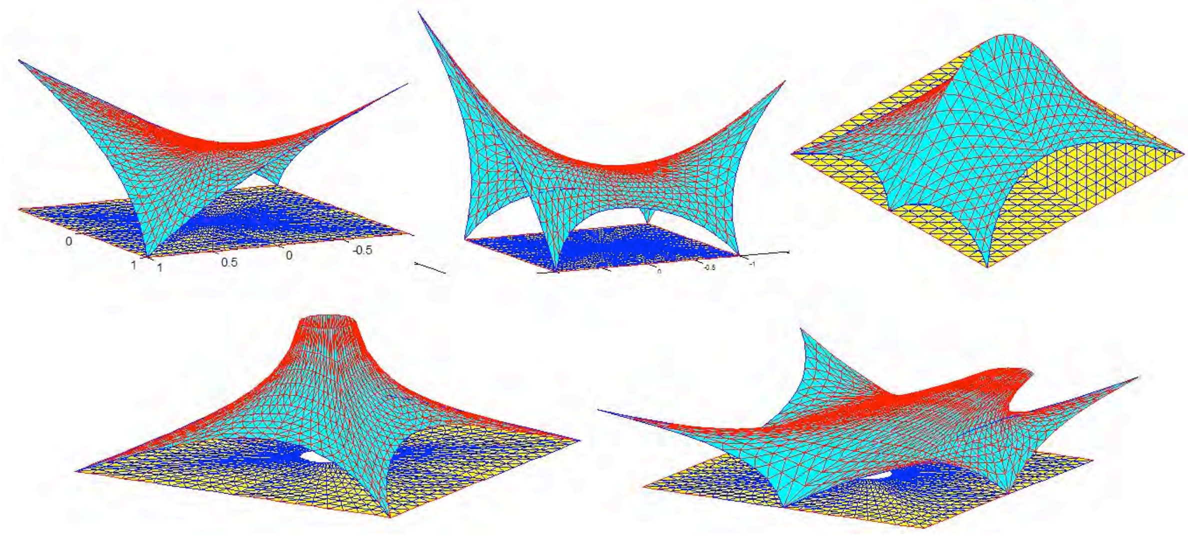

Figure 2: Viable configurations generated through the imposition of different sets of nodal displacements to the same plane squared reference mesh, with and without a hole.

As a second example, inspired by a physical experiment illustrated by Isenberg[8], consider a helicoidal soap film, shown in Fig. 3(a). The same previous square reference mesh is deformed, such that sides S1 and S2 are transformed into small radial segments (see Fig. 3(b/e)). Side S2 is displaced transversally to the reference plane. Side S3 is deformed into a helix. Side S4 is constrained to slip over the vertical axis. Fig. 3(b) shows the initial reference mesh and the resulting geometry, associated to a Cauchy stress field with quite high concentrations close to borders S2, S3 and S4. Subsequent iterations do not alter the geometry significantly, but do smooth the stress field. After the 10th iteration, a practically uniform, isotropic Cauchy stress field is achieved, with the 1st principal Cauchy stress (σ_I) ranging from 1.005 to 1.063. Thus, the minimal surface associated with the prescribed boundary is in practice obtained.

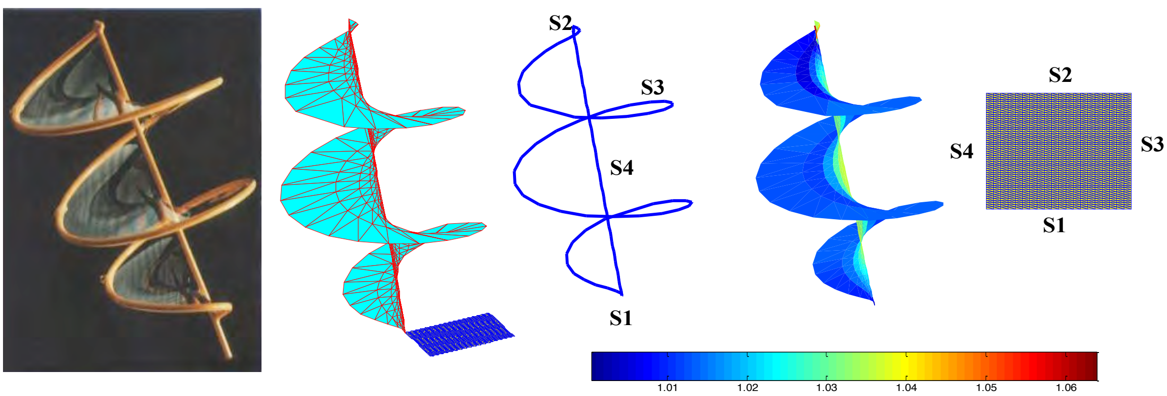

Figure 3: A helicoidal soap film

Next, we consider the generation of a minimal Costa's surface[9], starting from a non-minimal, non-smooth one (topologically, there is no distinction between them). In its 1st row, Fig. 4(a) shows a non-minimal Costa's surface connecting three fixed circular rings. Figs. 4(b/c) show the geometry obtained after the 1st and 6th iteration of the NFDM. It is seen that the 1st iteration of the NFDM already provides a fair approximation to the minimal surface. At the 2nd row, Figs. 4(d/e/f) show the σ_I fields resulting after the 1st iteration ($1.0288 \le \sigma_I \le 1.8086$), the 2nd iteration ($1.0015 \le \sigma_I \le 1.0594$) and the 6th iteration ($1.0001 \le \sigma_I \le 1.0124$). It is seen that after the 2nd iteration the σ_I field has already smoothed out any stress concentrations.

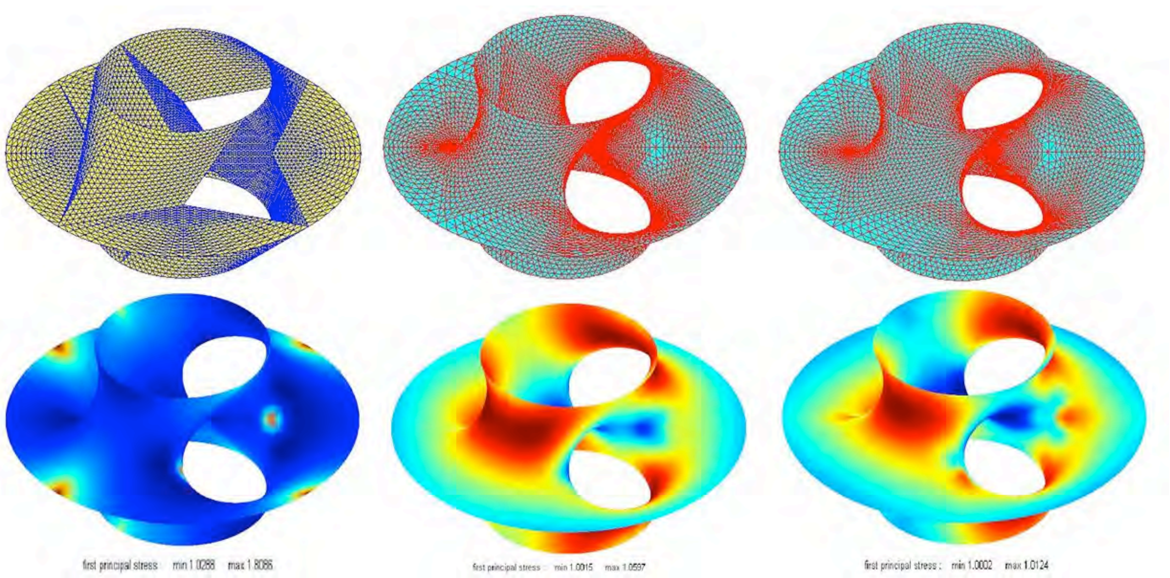

Figure 4: Numerical model of Costa's Surface

In Fig. 5(a) we show a physical realization of Costa's surface, exhibited at the atrium of the Civil Engineering building of the Polytechnic School of the University of São Paulo. Fig. 5(b) shows the patterning used to produce the physical model.

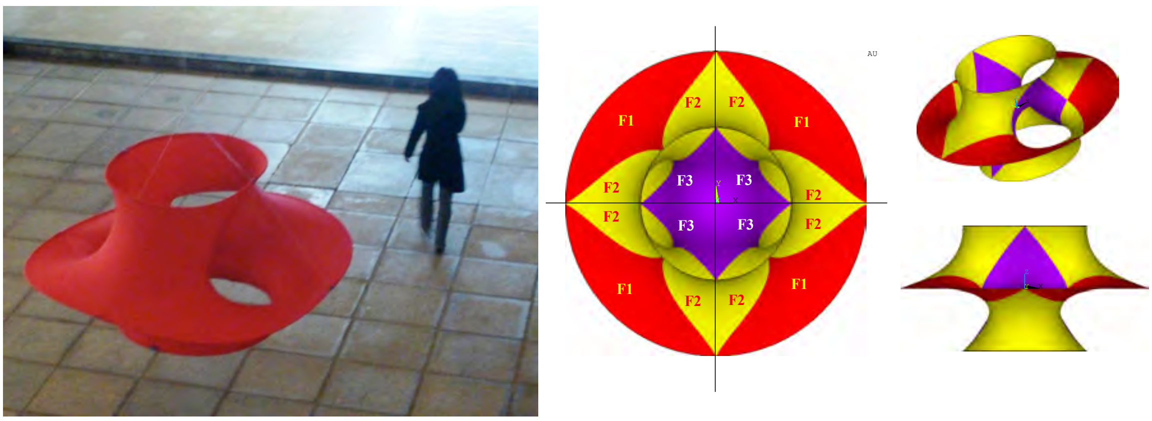

Figure 5: Physical model of Costa's Surface and corresponding fabric patterning

As a final application, Figure 6 compares a minimal conoid (stress ratio $\sigma_r / \sigma_\theta = 1$, necessarily, over the whole surface) to a non-minimal conoid ($\sigma_r / \sigma_\theta = 3$, arbitrary imposed over the whole surface). Both geometries were obtained after 10 NFDM iterations, required for convergence of the stress ratios. Results compare very well with analytical solutions, as shown in reference[10]. It is also important to point out that geometry converges much faster and, for practical purposes, the analysis could be stopped after a single iteration, or a couple of them, since there is no point in performing several iterations chasing a result (the imposed stress ratio) which is known *a priori*.

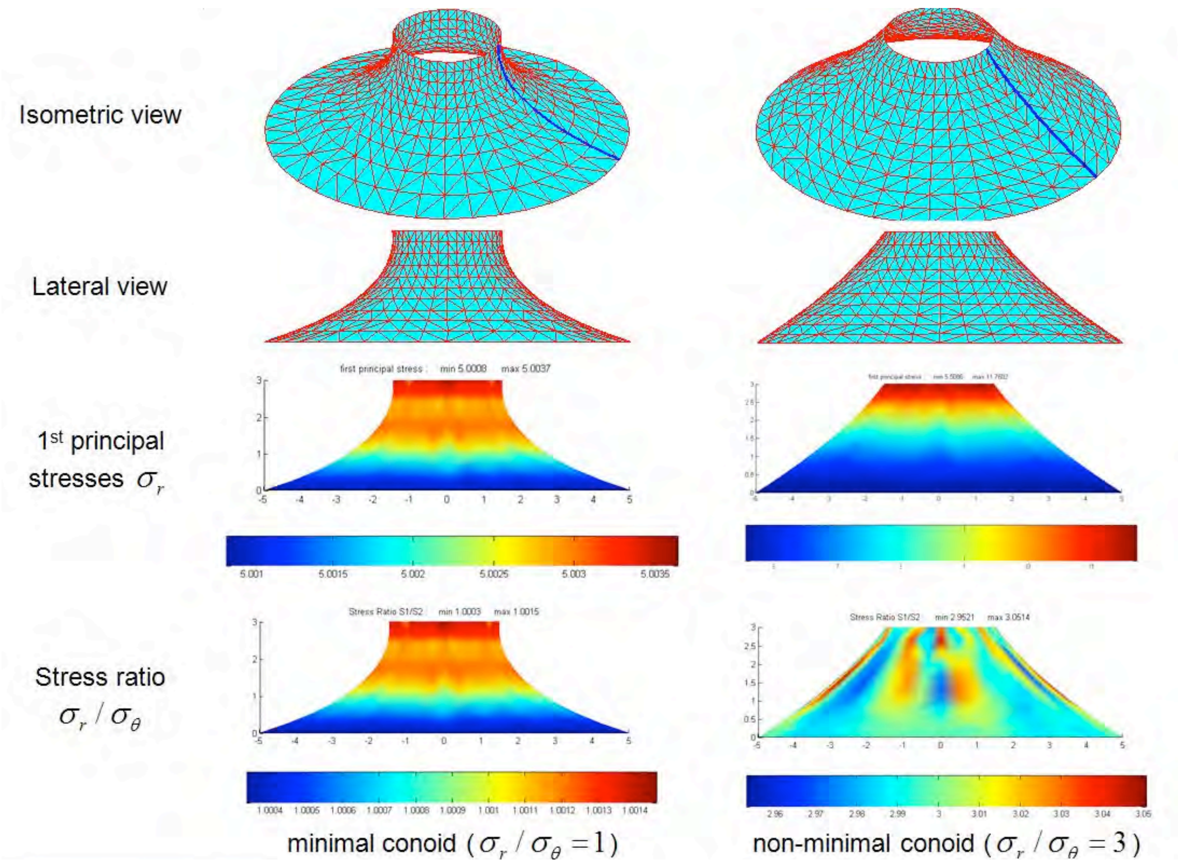

Figure 6. Comparison between minimal and non-minimal conoids.

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