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SHAPING OF NON-AXISYMMETRIC CONE STRUCTURES WITH NON-UNITY FIXED BI-AXIAL STRESS RATIO: CAN IT BE DONE?

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Abstract

A series of papers regarding the form finding of conic membrane structures with fixed stress ratio not equal to unity reported numerical instabilities in the convergence properties of the iterative procedures employed when the structure was not axisymmetric. This paper will find a theoretical basis for this instability and propose possible methods to obtain approximate solutions to the problem.

Keywords: Form finding, conic membrane structures, non-equal bi-axial stress

1. Introduction

In a series of papers dating back to the 2005 IASS Symposium [1 – 9], the authors have investigated the shaping of cone structures with fixed bi-axial stress ratios. The most important work in this series was an analytical study of determining the maximum separation allowable of an axisymmetric cone structure bounded by two circular rings of the same radius as a function of the stress ratio [8]. These results were verified using commercial software [9].

Some of these papers included numerical analyses for non-axisymmetric cone structures, particularly those bounded by elliptical rings [4 – 7, 9]. Generally, convergence properties of the analyses with these models were marked by an initial convergence to what appears to be a reasonable extent, followed by a divergence. Visually, the convergence procedure was marked by an eventual distortion of the mesh used in the finite element analysis. Various methods were tried to eliminate this problem by modifying the algorithm used in the iterative procedure. To date, none of these methods have worked particularly well. Most recently, the authors had to withdraw an accepted abstract for the 2013 IASS Symposium because of unsatisfactory results.

The authors eventually began to question the theoretical feasibility of attaining an equilibrium shape for non-axisymmetric cone structures with a fixed ratio between the meridianal and circumferential stresses other than unity. It was agreed that a "back to basics" investigation of the problem was required.

In this paper, the authors look at the problem using a principle of virtual work approach starting with the most basic principles of shell theory to investigate if it is possible to obtain at least approximate solutions analytically and/or numerically for this problem. As a first exercise, the paper will demonstrate that the method does lead to the identical formulation used in the original 2005 paper for axisymmetric cone structures. Then, the more general formulation will be discussed, identifying what may be the cause of prior difficulties in getting solutions to this problem when the stress ratio is not one. Finally, the general formulation will be employed to at least generate equations that can be used to possibly obtain approximate solutions that can be used to understand the behavior of these structures.

2. Formulation for Axisymmetric Structures

The formulation will be accomplished using cylindrical coordinates (r, φ, z). A point on the surface of the membrane is given by:

$$\vec{r}(z,\varphi) = r(z)\hat{e}_r + z\hat{e}_z \tag{1}$$

The φ dependence is implicit within the radial unit vector. The primary directional vectors are given by:

$$\begin{aligned}\vec{E}_1 &= \frac{\partial \vec{r}}{\partial z} = \frac{dr}{dz}\hat{e}_r + \hat{e}_z \\ \vec{E}_2 &= \frac{\partial \vec{r}}{\partial \varphi} = r\hat{e}_\varphi\end{aligned} \tag{2}$$

The "1" and "2" directions will be referred to as the meridianal and circumferential directions, respectively. The important thing to note is that these vectors are perpendicular to each other; as a result, they can serve as principal axes, and therefore it is possible that a bi-axial unequal stress state (with no shear stress) can exist in these directions, explaining the analytical and numerical successes obtained with axisymmetric structures.

To obtain unit vectors, one must divide the vectors of Eq. (2) by their respective magnitudes. These are:

$$\left|\vec{E}_1\right| = \sqrt{1+\left(\frac{dr}{dz}\right)^2};\left|\vec{E}_2\right| = r \tag{3}$$

Suppose virtual displacements $\delta u_r(z)$ and $\delta u_z(z)$ are added to the expression of Eq. (1). Deformed directional vectors are then given by:

$$\begin{aligned}\vec{E}_1{}' &= \left(\frac{dr}{dz} + \delta\frac{du_r}{dz}\right)\hat{e}_r + \left(1+\delta\frac{du_z}{dz}\right)\hat{e}_z \\ \vec{E}_2{}' &= (r+\delta u_r)\hat{e}_\varphi\end{aligned} \tag{4}$$

The virtual strains are given by:

$$\begin{aligned}\delta\varepsilon_1 &= \frac{1}{2}\left[\frac{\vec{E}_1{}' \cdot \vec{E}_1{}' - \vec{E}_1 \cdot \vec{E}_1}{\left|\vec{E}_1\right|^2}\right] \\ \delta\varepsilon_2 &= \frac{1}{2}\left[\frac{\vec{E}_2{}' \cdot \vec{E}_2{}' - \vec{E}_2 \cdot \vec{E}_2}{\left|\vec{E}_2\right|^2}\right]\end{aligned} \tag{5}$$

Substituting Eqs. (2 – 4) into Eq. (5) and ignoring powers of the virtual quantities higher than 1 will yield:

$$\begin{aligned}\delta\varepsilon_1 &= \frac{\dfrac{dr}{dz}\delta\dfrac{du_r}{dz} + \delta\dfrac{du_z}{dz}}{1+\left(\dfrac{dr}{dz}\right)^2} \\ \delta\varepsilon_2 &= \frac{\delta u_r}{r}\end{aligned} \tag{6}$$

The principle of virtual work is now applied to obtain equilibrium conditions. Introducing the meridianal and circumferential stress resultants yields:

$$\iint_S \left(S_1\delta\varepsilon_1 + S_2\delta\varepsilon_2\right)dS = 0 \tag{7}$$

The integral is over the membrane surface; the elemental surface area is given by:

$$dS = \left|\vec{E}_1\right|\left|\vec{E}_2\right|dzd\varphi \tag{8}$$

Substituting Eqs. (3) and (8) into Eq. (7), and noting that the integration over φ yields a multiplicative factor of 2π yields:

$$\int_z \left\{ \left(S_1 \frac{r \frac{dr}{dz}}{\sqrt{1+\left(\frac{dr}{dz}\right)^2}} \right) \delta \frac{du_r}{dz} + \left(S_1 \frac{r}{\sqrt{1+\left(\frac{dr}{dz}\right)^2}} \right) \delta \frac{du_z}{dz} + S_2 \sqrt{1+\left(\frac{dr}{dz}\right)^2}\, \delta u_r \right\} dz = 0 \tag{9}$$

Integrating by parts and, in the conventional way, stating that what multiplies each of the virtual displacements must be zero for equilibrium yields the two equations:

$$\frac{d}{dz}\left[S_1 \frac{r}{\sqrt{1+\left(\frac{dr}{dz}\right)^2}} \right] = 0 \tag{10}$$

$$S_2 \sqrt{1+\left(\frac{dr}{dz}\right)^2} - \frac{d}{dz}\left[S_1 \frac{r \frac{dr}{dz}}{\sqrt{1+\left(\frac{dr}{dz}\right)^2}} \right] = 0 \tag{11}$$

Carrying out the derivative in Eq. (11) yields:

$$S_2 \sqrt{1+\left(\frac{dr}{dz}\right)^2} - \frac{d}{dz}\left[S_1 \frac{r}{\sqrt{1+\left(\frac{dr}{dz}\right)^2}} \right] \frac{dr}{dz} - S_1 \frac{r}{\sqrt{1+\left(\frac{dr}{dz}\right)^2}} \frac{d^2 r}{dz^2} = 0 \tag{12}$$

The middle term in Eq. (12) is zero by Eq. (10); thus, Eq. (12) reduces to:

$$S_2 \sqrt{1+\left(\frac{dr}{dz}\right)^2} - S_1 \frac{r}{\sqrt{1+\left(\frac{dr}{dz}\right)^2}} \frac{d^2 r}{dz^2} = 0 \tag{13}$$

It is critical to note that in Eq. (13) that neither stress appears within a derivative expression. Declaring that the ratio of the meridianal stress to the circumferential stress is a fixed value α yields an equation for $r(z)$:

$$1+\left(\frac{dr}{dz}\right)^2 = \alpha r \frac{d^2 r}{dz^2} \tag{14}$$

In order to solve this equation, a substitution is made; defining:

$$y = \frac{dr}{dz} \tag{15}$$

and assuming that y is expressed as a function of r so that:

$$\frac{d^2 r}{dz^2} = y \frac{dy}{dr} \tag{16}$$

yields, after separation of variables:

$$\frac{dr}{r} = \alpha \frac{y}{1+y^2} dy \tag{17}$$

Integrating, and applying the boundary condition that at the neck of the conic structure $r = C$, the necking radius, while $y = 0$, and manipulating yields:

$$y = \sqrt{\left(\frac{r}{C}\right)^{\frac{2}{\alpha}} - 1} \tag{18}$$

Substituting back Eq. (15), separating variables, and setting z to zero at the neck yields:

$$z = \int_C^r \frac{dr}{\sqrt{\left(\frac{r}{C}\right)^{\frac{2}{\alpha}} - 1}} \tag{19}$$

Though derived in a totally new way, Eq. (19) is the same equation as used in [1] and [8]. This result verifies the approach used herein and offers clues to the more general case of non-axisymmetric membrane surfaces.

3. Formulation for Non-Axisymmetric Structures

The strategy here is to follow the procedure of Section 2 of this paper but extend it to more general conic membrane surfaces. The goals are to explain why the methodology as applied to non-axisymmetric structures does not work when the stress ratio is not unity, and how best to work around this fact to obtain equations to approximate as best as possible a shape for the membrane under somewhat similar conditions. The key to the second goal is to derive an equation similar to Eq. (13) above; that is, it is desired to have an equation where the stresses do not appear within derivatives and where the stress ratio can be worked in naturally, resulting in an equation for the shape in terms of the parameter α.

As before, a point on the membrane surface is defined by:

$$\vec{r} = r(z,\varphi)\hat{e}_r + z\hat{e}_z \tag{20}$$

Note that now r is a function of both z and φ. The primary directional vectors are defined as:

$$\begin{aligned} \vec{E}_1 &= \frac{\partial \vec{r}}{\partial z} = \frac{\partial r}{\partial z}\hat{e}_r + \hat{e}_z \\ \vec{E}_2 &= \frac{\partial \vec{r}}{\partial \varphi} = \frac{\partial r}{\partial \varphi}\hat{e}_r + r\hat{e}_\varphi \end{aligned} \tag{21}$$

Eq. (21) represents the crux of the problem. The two primary directional vectors are perpendicular to each other only where one of the partial derivatives of r is zero. For the cone structures with bi-planar symmetry studied in the past, the partial with respect to φ is only zero for the meridians in one of the primary planes. In general, for other locations (not counting the necking plane), the two vectors are not perpendicular and thus cannot form a set of principal axes. When the stress ratio is unity, all axes are principal axes and thus the fact that the meridianal and circumferential directions are not perpendicular has no deleterious effect. This explains why there are no numerical instabilities when the stress ratio is one, but why there exists numerical instability for ratios not equal to one, where there would be one and only one set of principal axes.

It should be noted that one could define a "local" meridian by finding the normal to the surface and taking a cross product of that vector with the circumferential vector. This was attempted in 2013 and lead to similarly unsatisfactory results.

From an exact equilibrium perspective, Eqs. (21) mark the end of the search for an equilibrium shape for general cone structures with fixed, non-unity stress ratios; however, the solutions found in [4], [5] and [7] appear reasonable and do converge partially before diverging. In order to understand this process, it is necessary to continue with the formulation begun herein. Depending on what is found, a path to approximate solutions may be determined.

The magnitudes of the primary directional vectors are:

$$\left|\vec{E}_1\right| = \sqrt{1+\left(\frac{\partial r}{\partial z}\right)^2}\, ; \left|\vec{E}_2\right| = r\sqrt{1+\left(\frac{1}{r}\frac{\partial r}{\partial \varphi}\right)^2} \tag{22}$$

For the general case, *three* virtual displacements are defined: δu_r, δu_z and δu_φ. Deformed directional unit vectors are thus calculated to be:

$$\begin{aligned} \vec{E}_1{}' &= \left(\frac{\partial r}{\partial z} + \delta\frac{\partial u_r}{\partial z}\right)\hat{e}_r + \delta\frac{\partial u_\varphi}{\partial z}\hat{e}_\varphi + \left(1+\delta\frac{\partial u_z}{\partial z}\right)\hat{e}_z \\ \vec{E}_2{}' &= \left(\frac{\partial r}{\partial \varphi} + \delta\frac{\partial u_r}{\partial \varphi} - \delta u_\varphi\right)\hat{e}_r + \left(r + \delta u_r + \delta\frac{\partial u_\varphi}{\partial \varphi}\right)\hat{e}_\varphi + \delta\frac{\partial u_z}{\partial \varphi}\hat{e}_z \end{aligned} \tag{23}$$

The virtual strains are defined as in Eq. (5) above; employing the same rules, the virtual strains are:

$$\begin{aligned} \delta\varepsilon_1 &= \frac{\dfrac{\partial r}{\partial z}\delta\dfrac{\partial u_r}{\partial z} + \delta\dfrac{\partial u_z}{\partial z}}{1+\left(\dfrac{\partial r}{\partial z}\right)^2} \\ \delta\varepsilon_2 &= \frac{\dfrac{\partial r}{\partial \varphi}\delta\dfrac{\partial u_r}{\partial \varphi} - \dfrac{\partial r}{\partial \varphi}\delta u_\varphi + r\delta u_r + r\delta\dfrac{\partial u_\varphi}{\partial \varphi}}{r^2\left[1+\left(\dfrac{1}{r}\dfrac{\partial r}{\partial \varphi}\right)^2\right]} \end{aligned} \tag{24}$$

Applying the principle of virtual work and employing the surface element defined by Eq. (8), one obtains:

$$\begin{aligned} \iint_S &\left\{ S_1\frac{r\dfrac{\partial r}{\partial z}\sqrt{1+\left(\dfrac{1}{r}\dfrac{\partial r}{\partial \varphi}\right)^2}}{\sqrt{1+\left(\dfrac{\partial r}{\partial z}\right)^2}}\delta\frac{\partial u_r}{\partial z} + S_1\frac{r\sqrt{1+\left(\dfrac{1}{r}\dfrac{\partial r}{\partial \varphi}\right)^2}}{\sqrt{1+\left(\dfrac{\partial r}{\partial z}\right)^2}}\delta\frac{\partial u_z}{\partial z} + S_2\frac{\dfrac{\partial r}{\partial \varphi}\sqrt{1+\left(\dfrac{\partial r}{\partial z}\right)^2}}{r\sqrt{1+\left(\dfrac{1}{r}\dfrac{\partial r}{\partial \varphi}\right)^2}}\delta\frac{\partial u_r}{\partial \varphi} \right. \\ &\left. - S_2\frac{\dfrac{\partial r}{\partial \varphi}\sqrt{1+\left(\dfrac{\partial r}{\partial z}\right)^2}}{r\sqrt{1+\left(\dfrac{1}{r}\dfrac{\partial r}{\partial \varphi}\right)}}\delta u_\varphi + S_2\frac{\sqrt{1+\left(\dfrac{\partial r}{\partial z}\right)^2}}{\sqrt{1+\left(\dfrac{1}{r}\dfrac{\partial r}{\partial \varphi}\right)^2}}\delta u_r + S_2\frac{\sqrt{1+\left(\dfrac{\partial r}{\partial z}\right)^2}}{\sqrt{1+\left(\dfrac{1}{r}\dfrac{\partial r}{\partial \varphi}\right)^2}}\delta\frac{\partial u_\varphi}{\partial \varphi} \right\} dz\,d\varphi = 0 \end{aligned} \tag{25}$$

This will lead to three equilibrium equations. First, in the axial direction:

$$\frac{\partial}{\partial z}\left[S_1\frac{r\sqrt{1+\left(\dfrac{1}{r}\dfrac{\partial r}{\partial \varphi}\right)^2}}{\sqrt{1+\left(\dfrac{\partial r}{\partial z}\right)^2}}\right] = 0 \tag{26}$$

Then, in the circumferential direction:

$$S_2 \frac{\frac{1}{r}\frac{\partial r}{\partial \varphi}\sqrt{1+\left(\frac{\partial r}{\partial z}\right)^2}}{\sqrt{1+\left(\frac{1}{r}\frac{\partial r}{\partial \varphi}\right)^2}} + \frac{\partial}{\partial \varphi}\left[S_2 \frac{\sqrt{1+\left(\frac{\partial r}{\partial z}\right)^2}}{\sqrt{1+\left(\frac{1}{r}\frac{\partial r}{\partial \varphi}\right)^2}}\right] = 0 \tag{27}$$

Finally, in the radial direction:

$$S_2 \frac{\sqrt{1+\left(\frac{\partial r}{\partial z}\right)^2}}{\sqrt{1+\left(\frac{1}{r}\frac{\partial r}{\partial \varphi}\right)^2}} - \frac{\partial}{\partial z}\left[S_1 \frac{r\frac{\partial r}{\partial z}\sqrt{1+\left(\frac{1}{r}\frac{\partial r}{\partial \varphi}\right)^2}}{\sqrt{1+\left(\frac{\partial r}{\partial z}\right)^2}}\right] - \frac{\partial}{\partial \varphi}\left[S_2 \frac{\frac{1}{r}\frac{\partial r}{\partial \varphi}\sqrt{1+\left(\frac{\partial r}{\partial z}\right)^2}}{\sqrt{1+\left(\frac{1}{r}\frac{\partial r}{\partial \varphi}\right)^2}}\right] = 0 \tag{28}$$

Carrying out the derivatives within Eq. (28) yields:

$$\begin{aligned} &S_2 \frac{\sqrt{1+\left(\frac{\partial r}{\partial z}\right)^2}}{\sqrt{1+\left(\frac{1}{r}\frac{\partial r}{\partial \varphi}\right)^2}} - \frac{\partial}{\partial z}\left[S_1 \frac{r\sqrt{1+\left(\frac{1}{r}\frac{\partial r}{\partial \varphi}\right)^2}}{\sqrt{1+\left(\frac{\partial r}{\partial z}\right)^2}}\right]\frac{\partial r}{\partial z} - S_1 \frac{r\sqrt{1+\left(\frac{1}{r}\frac{\partial r}{\partial \varphi}\right)^2}}{\sqrt{1+\left(\frac{\partial r}{\partial z}\right)^2}}\frac{\partial^2 r}{\partial z^2} \\ &-\frac{\partial}{\partial \varphi}\left[S_2 \frac{\sqrt{1+\left(\frac{\partial r}{\partial z}\right)^2}}{\sqrt{1+\left(\frac{1}{r}\frac{\partial r}{\partial \varphi}\right)^2}}\right]\frac{1}{r}\frac{\partial r}{\partial \varphi} - S_2 \frac{\sqrt{1+\left(\frac{\partial r}{\partial z}\right)^2}}{\sqrt{1+\left(\frac{1}{r}\frac{\partial r}{\partial \varphi}\right)^2}}\frac{\partial}{\partial \varphi}\left(\frac{1}{r}\frac{\partial r}{\partial \phi}\right) = 0 \end{aligned} \tag{29}$$

Substituting Eqs. (26) and (27) into the second and fourth terms of Eq. (29), and performing some manipulation yields:

$$S_2 \frac{\sqrt{1+\left(\frac{\partial r}{\partial z}\right)^2}}{\sqrt{1+\left(\frac{1}{r}\frac{\partial r}{\partial \varphi}\right)^2}}\left[1+2\left(\frac{1}{r}\frac{\partial r}{\partial \varphi}\right)^2 - \frac{1}{r}\frac{\partial^2 r}{\partial \varphi^2}\right] - S_1 \frac{r\sqrt{1+\left(\frac{1}{r}\frac{\partial r}{\partial \varphi}\right)^2}}{\sqrt{1+\left(\frac{\partial r}{\partial z}\right)^2}}\frac{\partial^2 r}{\partial z^2} = 0 \tag{30}$$

Eq. (30) is in the form conducive to possibly finding solutions because neither of the stresses now appears within a derivative expression. Dividing Eq. (30) by S_2 yields a partial differential equation for $r(z,\varphi)$:

$$\frac{1+2\left(\frac{1}{r}\frac{\partial r}{\partial \varphi}\right)^2 - \frac{1}{r}\frac{\partial^2 r}{\partial \varphi^2}}{1+\left(\frac{1}{r}\frac{\partial r}{\partial \varphi}\right)^2} = \frac{\alpha r \frac{\partial^2 r}{\partial z^2}}{1+\left(\frac{\partial r}{\partial z}\right)^2} \tag{31}$$

For structures with bi-planar symmetry, the appropriate boundary conditions are:

$$\frac{\partial r}{\partial z}(0,\varphi) = 0; r(h/2,\varphi) = g(\varphi); \frac{\partial r}{\partial \varphi}(z,0) = \frac{\partial r}{\partial \varphi}(z,\pi/2) = 0 \tag{32}$$

Here, h is the separation between two identical rings and the shape of the rings, $g(\varphi)$, is somewhat arbitrary to the extent that it satisfies the last two boundary conditions.

4. Possible Paths to Solution

Two approaches to solve the system of Eqs. (31) and (32) are suggested in this section.

The first approach takes its cue from the results of [9]. In that paper, which assumed an elliptical shape for the boundary rings, the solution appeared to be a summation of the solution for circular rings of radius equal to the geometric mean radius of the ellipse plus a correction which accounted for the elliptical shape at the ring; furthermore, at the mid-plane, the membrane took on an elliptical shape as well but with a smaller aspect ratio than that of the boundary rings. The impact of these results will be incorporated below.

Looking at the circumferential boundary conditions, it is not unreasonable to assume that $r(z,\varphi)$ could be written as:

$$r(z,\varphi) = \sum_{n=0}^{\infty} Z_{2n}(z)\cos 2n\varphi \tag{33}$$

The remaining boundary conditions can be expressed as:

$$\frac{dZ_{2n}}{dz}(0) = 0; \sum_{n=0}^{\infty} Z_{2n}(h/2)\cos 2n\varphi = g(\varphi) \tag{34}$$

The form of Eq. (33) can be substituted into Eq. (31). The φ dependence is rather complex after the substitution. To alleviate this situation, the equation can be multiplied by cos $2n\varphi$ and integrated over the range of φ. If a finite number of terms in Eq. (33) is assumed (say, 2), then there will be two equations with two unknowns which have a chance of being solved either analytically or numerically. It may be useful to initialize Z_0 to be in a form identical to that of the axisymmetric solution with, say, a value of the average ring radius at $h/2$. This may lead to an easier approximate approach for solving for Z_2.

The second approach takes its cue from the substitution made in the axisymmetric solution in Eq. (15). First, Eq. (31) is re-written as:

$$1 - \frac{\frac{\partial}{\partial\varphi}\left(\frac{1}{r}\frac{\partial r}{\partial\varphi}\right)}{1+\left(\frac{1}{r}\frac{\partial r}{\partial\varphi}\right)^2} = \frac{\alpha r\frac{\partial^2 r}{\partial z^2}}{1+\left(\frac{\partial r}{\partial z}\right)^2} \tag{35}$$

The following variables are defined:

$$y = \frac{\partial r}{\partial z}; q = \frac{1}{r}\frac{\partial r}{\partial\varphi} \tag{36}$$

Both y and q are assumed to be explicit functions of *r and* φ. Eqs. (36) are substituted into Eq. (35) to yield:

$$1 - \frac{\frac{\partial q}{\partial\varphi}}{1+q^2} = \frac{\alpha r y}{1+y^2}\frac{\partial y}{\partial r} \tag{37}$$

Noting the form of Eqs. (36), a second equation is derived as:

$$\frac{\partial y}{\partial\varphi} = \frac{\partial}{\partial z}(rq) = \frac{\partial}{\partial r}(rq)\frac{\partial r}{\partial z} = y\frac{\partial}{\partial r}(rq) \tag{38}$$

After some re-arrangement and manipulation, Eq. (38) becomes:

$$\frac{\partial}{\partial\varphi}(\ln y) = \frac{\partial}{\partial r}(rq) \tag{39}$$

Perhaps an iterative procedure alternating between Eqs. (37) and (39) would yield fruitful results.

5. Conclusions

It is now conceded by the authors that formulations using natural definitions of the meridianal and circumferential coordinates for non-axisymmetric conic structures lead to directions which are not perpendicular and thus cannot have a stress state indicative of principal axes unless the two normal stresses are equal. The formulation derived herein indicates that approximate solutions which can be used as guidelines in the design of conic membrane structures may be obtainable by mixed analytical and numerical means. Future research into this area will concentrate on obtaining these approximate solutions and/or add additional rigor to the formulation.

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